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1988 J. Phys. A: Math. Gen. 21 2261

(http://iopscience.iop.org/0305-4470/21/9/038)

J. Phys. A: Math. Gen. 21 (1988) 2261-2263. Printed in the UK

COMMENT

Spectral dimension of Sierpinski gasket type fractals

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Received 17 December 1987

Abstract. An explicit formula for the spectral dimension of the Sierpinski type fractal family studied by Borjan *et al* is obtained using the method of images. For large values of their parameter b, the spectral dimension $\tilde{d}(b) = 2 - (\log\log b)/\log b +$ terms of order $(1/\log b)$.

In a recent letter Borjan *et al* (1987) have determined the spectral dimension $\tilde{d}$ of an infinite class of Sierpinski type fractals by a repeated use of the star-triangle transformation. The fractals are characterised by an integer b. Borjan *et al* determined the spectral dimension of fractals with b varying from 2 to 200, but were not able to determine the asymptotic behaviour of $\tilde{d}(b)$ for large b from their data.

In this comment, we point out that an explicit formula for $\tilde{d}(b)$ can be obtained for arbitrary b using the method of images. Consider an electrical resistance network obtained by joining unit resistors so that they form the bonds of an infinite two-dimensional hexagonal lattice (figure 1). We set up a steady electric current in this network by keeping some of the nodes (called source sites) at a potential $+\Phi_0$, while

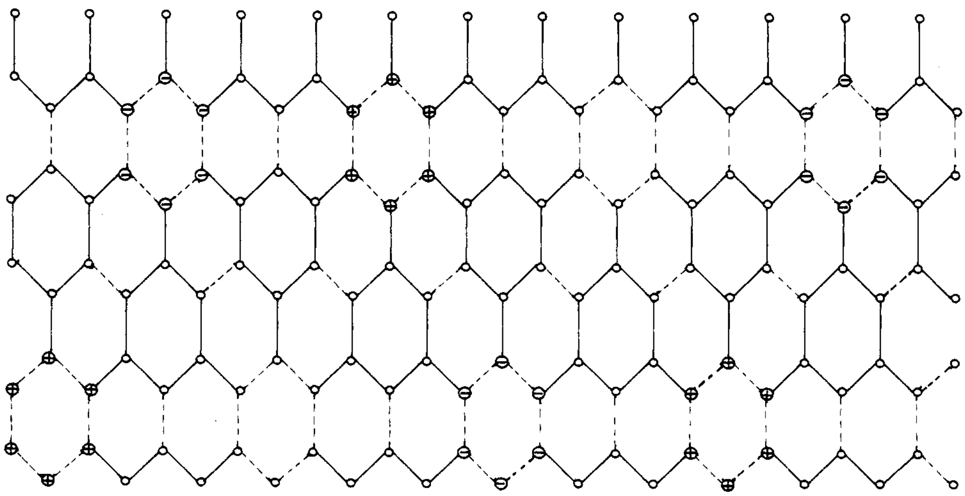

Figure 1. A portion of the two-dimensional electrical network made by joining unit resistors into an infinite hexagonal lattice. The current source sites (+) kept at a potential $+\Phi_0$ and the current sinks (−) kept at a potential $-\Phi_0$ are arranged periodically on the lattice. The figure shows the arrangement of (+) and (−) nodes for $b = 2$. The bonds shown by broken lines carry no current.

other nodes (called sink sites) are kept at potential $-\Phi_0$ by connecting them to an external voltage source. These 'source' and 'sink' nodes are arranged periodically on the lattice so as to form a hexagonal superlattice (figure 1). For every hexagon with all nodes at potential $+\Phi_0$, there are three nearby hexagons symmetrically placed at a distance $(b+1)$ (here b is any positive integer) lattice units with all vertices at potential $-\Phi_0$; and vice versa.

From the symmetry of the problem, it is quite clear that links shown by broken lines in figure 1 carry no current. These may be cut without affecting the current distribution in the rest of the lattice. If all such bonds are cut, the network breaks up into disconnected triangular-shaped finite networks, each of which is isomorphic to the generator of the b fractal studied by Borjan *et al* (1987) (figure 2) by a star-triangle transformation. The solution of the circuit equations for this generator is easily written down using the periodicity of the extended problem.

We choose a unit cell as shown in figure 3; there are two sites per unit cell. These will be labelled α and β. Let $\phi_\alpha(\bar{R})$ and $\phi_\beta(\bar{R})$ be the potentials at α and β sites respectively in the unit cell labelled by the two-dimensional integer vector $\bar{R}$. Then Kirchoff's equations for the network are

$$\nabla^2\begin{bmatrix}\phi_\alpha(R)\\ \phi_\beta(R)\end{bmatrix} = -\begin{bmatrix}\rho_\alpha(R)\\ \rho_\beta(R)\end{bmatrix} \tag{1}$$

where ∇^2 is the discrete Lapacian operator, and $\rho_\alpha(\bar{R})$ and $\rho_\beta(\bar{R})$ are the total current outflows from the sites α and β in the $\bar{R}$th cell. These are periodic in $\bar{R}$, and their Fourier expansions have only a finite number of terms. For the case where the current

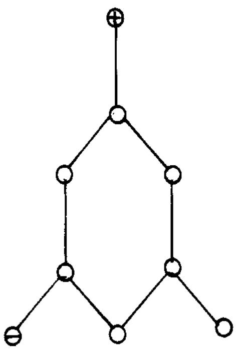

Figure 2. One of the finite networks obtained by removing the bonds denoted by broken lines in figure 1. This is related to the generator of the $b=2$ fractal by a star-triangle transformation.

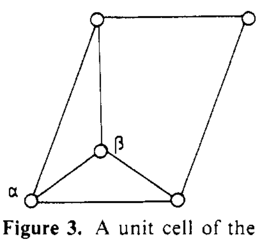

Figure 3. A unit cell of the hexagonal lattice lattice showing the labelling of sites within the unit cell.

out of each source site is +1, it is easy to see that

$$\begin{bmatrix} \rho_\alpha(\bar{R}) \\ \rho_\beta(\bar{R}) \end{bmatrix} = \sum_{\bar{k}} \exp(\mathrm{i}\bar{k}\cdot\bar{\boldsymbol{R}}) \frac{\{1-\exp[-\mathrm{i}k_1(b+1)]\}}{3(b+1)^2} \begin{bmatrix} 1+\exp(\mathrm{i}k_2)+\exp(\mathrm{i}k_2-\mathrm{i}k_1) \\ 1+\exp(-\mathrm{i}k_1)+\exp(\mathrm{i}k_2-\mathrm{i}k_1) \end{bmatrix} \tag{2}$$

where the summation over $\bar{k}=(k_1, k_2)$ extends over all values given by

$$k_1 = 2\pi m/3(b+1) \qquad \text{with} \qquad 1 \leqslant m < 3(b+1) \tag{3a}$$

$$k_1 + k_2 = 2\pi n/(b+1) \qquad \text{with} \qquad 0 \leqslant n < (b+1). \tag{3b}$$

In the momentum basis, the Laplacian is a diagonal operator, and we get the Fourier coefficients of the potential $\tilde{\phi}_\alpha(k)$ and $\tilde{\phi}_\beta(k)$ given by

$$\begin{bmatrix} \tilde{\phi}_\alpha(k) \\ \tilde{\phi}_\beta(k) \end{bmatrix} = \begin{bmatrix} 3 & -[1+\exp(\mathrm{i}k_1)+\exp(\mathrm{i}k_2)] \\ -[1+\exp(-\mathrm{i}k_1)+\exp(-\mathrm{i}k_2)] & 3 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\rho}_\alpha(k) \\ \tilde{\rho}_\beta(k) \end{bmatrix}. \tag{4}$$

But $\phi_\alpha(\bar{R}=0) = \phi_\beta(\bar{R}=0) = R_b$, the resistance scaling function in the notation of Borjan *et al* (1987). Using this fact from (2) and (4) we get

$$\begin{aligned} \begin{bmatrix} R_b \\ R_b \end{bmatrix} = \sum_{\bar{k}} \frac{\{1-\exp[-\mathrm{i}k_1(b+1)]\}}{3(b+1)^2} & \\ \times \begin{bmatrix} 3 & -[1+\exp(\mathrm{i}k_1)+\exp(\mathrm{i}k_2)] \\ -[1+\exp(-\mathrm{i}k_1)+\exp(-\mathrm{i}k_2)] & 3 \end{bmatrix}^{-1} & \\ \times \begin{bmatrix} 1+\exp(\mathrm{i}k_2)+\exp(\mathrm{i}k_2-\mathrm{i}k_1) \\ 1+\exp(-\mathrm{i}k_1)+\exp(\mathrm{i}k_2-\mathrm{i}k_1) \end{bmatrix} & \end{aligned} \tag{5}$$

where the summation over $\bar{k}$ again extends over all values given by equation (3). We have used the convention that $R_1 = 1$.

Equation (5) gives an explicit expression for R_b as a finite summation. Using the symmetries of the lattices, the number of terms appearing in the sum can be reduced somewhat. It is easy to verify that for $b = 1, 2, \ldots,$ etc, the formula agrees with the results of Borjan *et al* (1987). For large b, it follows from (5) that

$$R_b \sim K \log b \tag{6}$$

where K is some constant. This implies that

$$\tilde{d}(b) = 2 - \frac{\log\log b}{\log b} + \text{terms of order } (1/\log b)$$

for large b. This asymptotic behaviour has been verified numerically by Milošević *et al* (1988) using values of $b \leqslant 650$.

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